Duality between 2+1-D Transverse Field Ising Model (TFIM) & 2+1-D Ising Guage theory.

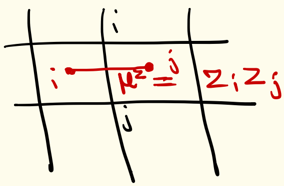
The O(2) model duality suggests that phases without local order parameter (e.g. the photon phase) may be understood more clearly in terms of dual varibles. Lets consider another example:

$$H = -J \sum_{\langle ij \rangle} s_i^z s_j^z - h \sum_i s_i^x$$

on square lattice. This is very similar to the O(2) model with $Z \sim e^{i\theta}$, $X \sim n^2$.

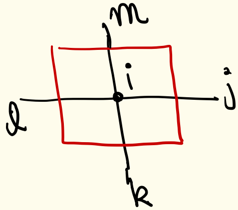
Lets define dual variables anologously:

$$s_i^z s_j^z = X_{ij}$$



where ij on the RHS refers to sites of the dual lattice.

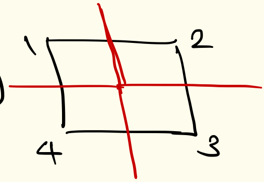
Similarly, $s_i^x = Z_{ij} Z_{ik} Z_{il} Z_{im}$



One can readily check that the commutation relations are satisfied: $X_{ij} Z_{ij} = -Z_{ij} X_{ij}$

Further, since

$$(S_1^z S_2^z)(S_2^z S_3^z)(S_3^z S_4^z)(S_4^z S_1^z)$$



$$\Rightarrow \quad = 1 = X_{12} X_{23} X_{34} X_{41}$$

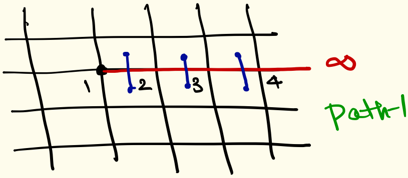
Interperating $\mu^x \sim e^{iE}$, $\mu^z \sim e^{iA}$, the above constraint corresponds to $\nabla \cdot E = 1$ (Gauss's law) and the Hamiltonian is a $\mathbb{Z}_2$

Gauge theory :

$$H = \underbrace{-J \sum X_{ij}}_{\substack{\equiv H_x \\ \sim e^{iE}}} - h \underbrace{\sum_{\square} \prod_{\square} Z_{ij}}_{\substack{\equiv H_P \\ \cos(\nabla \times A)}}$$

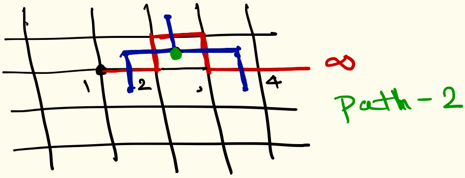
with $\prod_{+} X_{ij} = 1$ (Gauss's law $\nabla \cdot E = 0$)

One may also express the original spin variables S_i in terms of gauge theory variables :



$$\tilde{S}^z_1 = S^z_1 S^z_2 S^z_2 S^z_3 S^z_3 S^z_4 \cdots = X_{12} X_{23} X_{34} \cdots = \prod X_{ij}$$

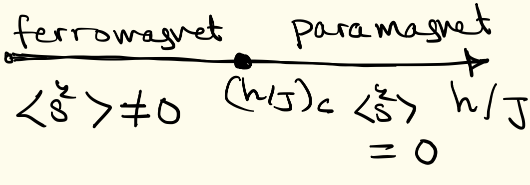
One may wonder if the path matters, since there are infinite many choices. The answer is no, due to the constraint $\prod_{+} X_{ij} = 1$.



e.g. path-2 can be obtained from path-1 by acting with $1 = \prod X_{ij}$ on site colored green.

Phase Diagram:

In terms of original spin-variables, the phase diagram is determined straight-forwardly:



This implies that the dual guage theory must also have two distinct phases. However, since S^z is a non-local variable in terms of $\{X, Z\}$, there is no local order-parameter in the guage theory. How does one then characterize the two phases of the guage theory?

Ans: Most fundamentally, in the guage theory Hamiltonian, the phase at $h/J \gg 1$ has topological order characterized by presence of anyons as excitations above the ground state, and ground state degeneracy that depends on the topology of the manifold. It also has long-range entanglement.

In contrast, the guage theory Hamiltonian at $h/J \ll 1$ corresponds to a phase without any topological order. Due to sharp diffences, these two phases are necessarily separated by a quantum phase transition.

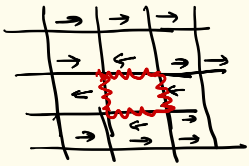
Ground state in the limit $h/J \ll 1$:

First consider $h = 0$. The ground state is

$|\psi_0\rangle = \prod_i |\rightarrow\rangle_i$ i.e. all spins point along $+\hat{x}$ direction.

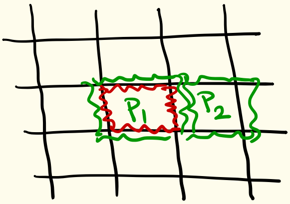
(This state automatically satisfies the Gauss's law)

This is a unique ground state on any manifold. (e.g. a sphere or a torus). The lowest energy excited state, consistent with the Gauss's law is obtained by flipping four spins from $+\hat{x}$ to $-\hat{x}$:



. This costs an energy $+8J$. These excitations can be located anywhere, so there is a degeneracy $\propto N$ where N is the number of sites.

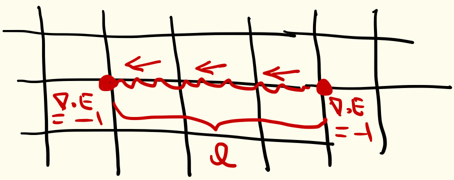
Turning on the h term, these excitations will acquire kinetic energy. A single action of $h\prod_\square \sigma^z$ creates an additional such excitation, so is prohibitely costly when $J \gg h$. One has to go to second order in perturbation theory to remain within the subspace of excitations with energy $8J$:



Action of $\prod_{P_1} Z \; \prod_{P_2} Z$ moves the excitation P_1 rightwards to P_2.

Therefore the low energy Hamiltonian for the excitations is: $-\frac{h^2}{J} \prod \sigma^z$ where denotes a rectangular loop of size 1×2 or 2×1.

Another kind of excitation one can define is two test charges created by $\prod \sigma^z$:



To allow this excitation, we have relaxed the Gauss's law at the end points of the string.

This excitation costs energy $2Jl$ where l is the separation between charges.

$\Rightarrow$ Charges are 'confined' in this phase. They do not propagate freely.

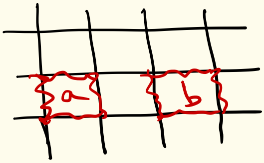
Derivation:

let's denote the state corresponding to a single excitation localized on plaquette 'x' as $|x\rangle$.

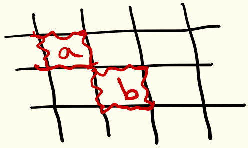
The effective Hamiltonian H_{eff} within the subspace of single excitations is given by

$$\langle a| H_{eff} |b\rangle = \langle a | H_p | b\rangle + \frac{\langle a | H_p | \mu\rangle \langle \mu | H_p | b\rangle}{E_a - E_\mu} + \dots$$

where $H_p = -J \sum_{\square} \prod_{\square} Z_i$ is the plaquette term, & $|\mu\rangle$ is a state that does not lie within the subspace of single excitations. The first order term $\langle a | H_p | b\rangle$ clearly vanishes. The nature of second order term depends on whether the plaquettes 'a', 'b' share an edge or not. First consider the case where they do not share an edge:



or



There are two different kinds of state $|\mu\rangle$. $|\mu\rangle$ can either be the ground state $|0\rangle$, or an excited state with two excitations. For the matrix element at second-order to be non-zero, this excited state must be $|a,b\rangle$ i.e. a state that has excitations at both a and b. The energy difference $E(|a\rangle) - E(|a,b\rangle)$ $= -8J$.

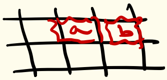
$$\Rightarrow \langle a | H_{eff} | b \rangle$$

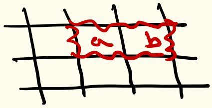
$$= \frac{\langle a | -h \prod_{\square\in a} Z - h \prod_{\square\in b} | a,b \rangle \times \langle a,b | -h \prod_{\square\in a} Z - h \prod_{\square\in b} | b \rangle}{-8J}$$

$$+ \frac{\langle a | -h \prod_{\square\in a} Z - h \prod_{\square\in b} | 0 \rangle \times \langle 0 | -h \prod_{\square\in a} Z - h \prod_{\square\in b} | b \rangle}{8J}$$

$$= \frac{h^2}{8J} - \frac{h^2}{8J} = 0 .$$ Perfect cancellation.

Therefore, this term vanishes. In fact, on general grounds we expect that when a, b are separated by distance l, one would need to go to $l+1$ order in perturbation theory.

Now, let's consider the case when a, b sit next to each other: . One of the intermediate states continue to be the ground state $|0\rangle$, and its contribution to the matrix element $\langle a | H_{eff} | b \rangle$ is unchanged.

But the other intermediate state now is a 'merger' of $|a\rangle$, $|b\rangle$: .

The energy diff. $E_a - E_\mu = 2 \times (4-6) J$
$= -4J$.

Thus, the cancellation is avoided.

$$\Rightarrow \langle a | H_{eff} | b \rangle = -\frac{h^2}{4J} + \frac{h^2}{8J} = -\frac{h^2}{8J}$$

when a, b share an edge, and zero otherwise.

Quasiparticles and their statistics :

The excitations of the form are the only well defined quasiparticles when $J \gg h$. They will form a band with dispersion $\varepsilon_k \sim J - \frac{2h^2}{8J}\left[\cos(k_x) + \cos(k_y)\right]$

When h increases, they may condense leading to a quantum phase transition although this leading order dispersion is not reliable when $h \sim J$.

What about the statistics of these excitations. One can ask two questions :

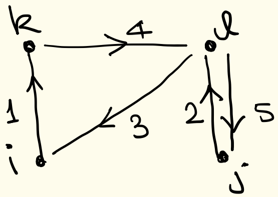
(a) Statistical Berry phase when two excitations are exchanged.

(b) Statistical Berry phase when a quasiparticle is taken around a test charge .

Since charges are confined, this isn't very physical. So, we will only consider (a).

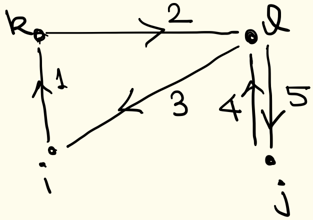
Following Levin, Wen (2003), to calculate the statistical Berry phase one needs to compare Berry phase between two different paths. Denoting t_{ij} as the operator that moves q.p. from j to i:

process ① $\hat{t}_{jl}\,\hat{t}_{lk}\,\hat{t}_{il}\,\hat{t}_{lj}\,\hat{t}_{ki}\,|i,j,\ldots\rangle$



Particles exchanged

process ② $\hat{t}_{jl}\,\hat{t}_{lj}\,\hat{t}_{il}\,\hat{t}_{lk}\,\hat{t}_{ki}\,|i,j,\ldots\rangle$

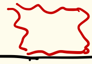


Particles <u>not</u> exchanged.

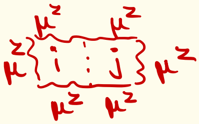
We assume there are no quasi particles at sites k, l in the initial state $|i,j,\ldots\rangle$.

The relative phase acquired by the wave-f^n between ① and ② is the statistical Berry phase (= exchange statistics). For fermions, this phase would be -1, and for bosons 1 (and $e^{i\theta}$ for abelian anyons, where θ can be any angle).

We are now in a position to answer the question we posed.

<u>Self-statistics of quasiparticles</u>  :

As discussed above the analog of operator $\hat{t}_{ij}$ that moves a quasiparticle from j to i is



. Since all μ_z's commute with each other $\Rightarrow$ No Berry phase difference between the two paths above $\Rightarrow$ statistical Berry phase $= 0$ $\Rightarrow$ excitations are bosons.